MATHEMATISCH CENTRUM
AMSTERDAM

ma
the
ma
tisch

cen
trum

AFDELING ZUIVERE WISKUNDE ZN 57/74 FEBRUARY

J. VAN DE LUNE

A NOTE ON THE FUNDAMENTAL THEOREM FOR RIEMANN-INTEGRALS

Prepublication

BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

amsterdam 1974

stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZN 57/74 FEBRUARY

J. VAN DE LUNE
A NOTE ON THE FUNDAMENTAL THEOREM
FOR RIEMANN-INTEGRALS

Prepublication

BIBLIOTHEEK
AMSTERDAM

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

A NOTE ON THE FUNDAMENTAL THEOREM FOR RIEMANN-INTEGRALS *)

by

Jan van de Lune

Usually the fundamental theorem for R-integrals

$$\int_a^b f'(x)dx = f(b) - f(a)$$

is proved under the assumption that f' is R-integrable over $[a,b]$. Compare, for example, [2, pg.115].

It appears that there is no answer in the literature to the question under what conditions the derivative f' of a differentiable function f is R-integrable (without making use of specific measure theoretic notions). Compare, for example, [1, pg.47].

In this note we prove the following

THEOREM. If $f : [a,b] \to \mathbb{R}$ is differentiable then the derivative f' is R-integrable over $[a,b]$ if and only if there exists an R-integrable function $\phi : [a,b] \to \mathbb{R}$ such that

$$f(x) = f(a) + \int_a^x \phi(t)dt.$$

In order to prove this theorem we need two lemmas.

Lemma 1. Let $\phi : [a,b] \to \mathbb{R}$ be R-integrable such that

$$m \leq \phi(x) \leq M, \qquad \forall x \in [a,b].$$

Define

$$\Phi(x) = \int_a^x \phi(t)dt, \qquad \forall x \in [a,b],$$

---

*) This paper is not for review; it is meant for publication in a journal.

and assume that Φ is differentiable on $[a,b]$.
Then we have

$$m \leqq \Phi'(x) \leqq M, \qquad \forall x \in [a,b].$$

Proof. Define $K(x) = \Phi(x) - m \,.\, (x-a)$.
Then $K(x)$ is differentiable such that

$$K'(x) = \Phi'(x) - m.$$

We also have

$$K(x) = \int_a^x \phi(t)dt - m \,.\, (x-a) = \int_a^x (\phi(t) - m)dt.$$

Since $\phi(t) \geqq m$ we see that K is monotonically non-decreasing. Hence

$$K'(x) \geqq 0,$$

or

$$\Phi'(x) \geqq m, \qquad \forall x \in [a,b].$$

In a similar fashion one proves

$$\Phi'(x) \leqq M, \qquad \forall x \in [a,b].$$

Lemma 2. Under the same conditions as in Lemma 1 we have that Φ' is R-integrable over $[a,b]$.

Proof. Lemma 1 is applicable on every closed subinterval of $[a,b]$. From this it follows that the fluctuation of Φ' is not larger than the fluctuation of ϕ.
According to a well known criterion for R-integrability [2, pg.107] we may conclude that Φ' is R-integrable over $[a,b]$.

Proof of the theorem. It is clear that the given condition is necessary; take $\phi = f'$.

To prove the sufficiency we write

$$f(x) - f(a) = \int_a^x \phi(t)dt.$$

Then it is clear that f' is the derivative of a function of the form

$$\int_a^x \phi(t)dt.$$

According to Lemma 2 we obtain that f' is R-integrable over [a,b], completing the proof.

References

[1] H. Kestelman, Modern theories of integration, Clarendon Press, Oxford, 1937.

[2] W. Rudin, Principles of mathematical analysis, McGraw-Hill, 1964.